GOSFORD HIGH SCHOOL

2009

Trial HSC

MATHEMATICS EXTENSION 2

Time Allowed: 3 Hours + 5 minutes reading time

General Instructions:

- Reading Time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.
- Each question should be started in a separate writing booklet.

TOTAL MARKS – 120

- Attempt Questions 1 – 8
- All questions are of equal value.

QUESTION 1: (Use a separate Writing Booklet)

(a) Evaluate $\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx$ (3)

(b)

(i) Use the substitution $x=\frac{\pi}{2}-u$ to show that:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,dx$$ (2)

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$. (2)

(c)

(i)Express $\frac{8}{(x+2)(x^2+4)}$ in the form $\frac{A}{x+2}+\frac{Bx+C}{x^2+4}$ (2)

(ii) Show that $\int_0^2 \frac{8}{(x+2)(x^2+4)}\,dx = \frac{1}{2}\log_e 2 + \frac{\pi}{4}$ (2)

(d) Given that $I_n = \int_0^1 x^n e^x\,dx$ for $n=0,1,2,...$

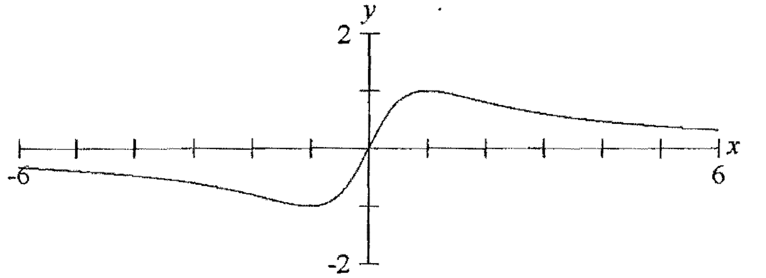
(i) Find I_0. (1)

(ii) Find an expression for I_n in terms of I_{n-1} for $n=0,1,2,...$ (2)

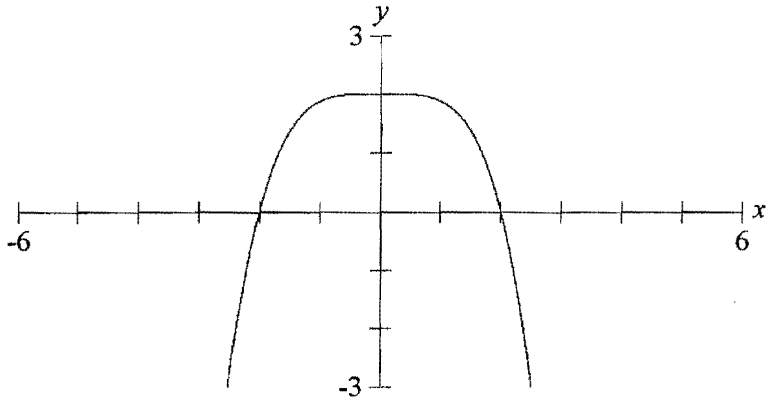
(iii) Evaluate I_3. (1)

QUESTION 2: (Use a separate Writing Booklet)

(a) The diagrams below represent the curves $f(x)=\dfrac{2x}{x^2+1}$ *and* $g(x)=2-\dfrac{x^4}{8}$.



$f(x)=\dfrac{2x}{x^2+1}$



$g(x)=2-\dfrac{x^4}{8}$

Use these diagrams to sketch the following functions (without calculus) showing all essential features.

(i) $y=f(-x)$ (1)

(ii) $y=|f(x)|$ (1)

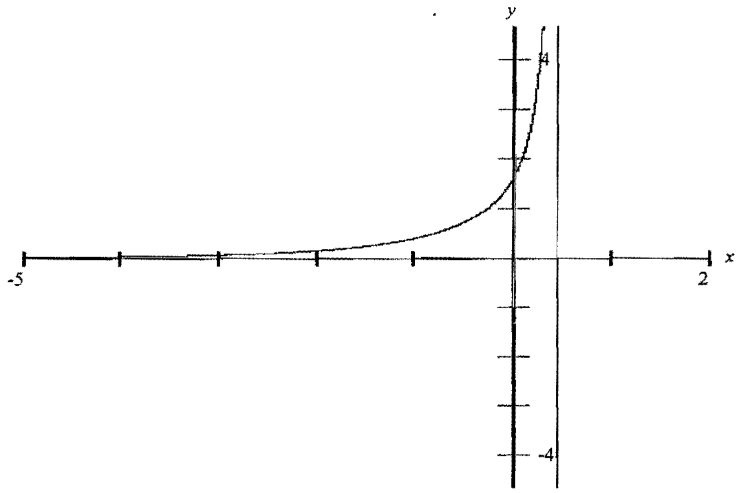
(iii) $y=\sqrt{f(x)}$ (1)

(iv) $y=\dfrac{1}{g(x)}$ (2)

(v) $y=[g(x)]^2$ (2)

(b) The diagram below shows part of the curve $y = \tan(e^x)$ where $x \langle \log_e(\frac{\pi}{2})$.

The part to the right has not yet been drawn.



$x = \log_e(\frac{\pi}{2})$

(i) By considering values of x greater than $x = \log_e(\frac{\pi}{2})$ find the smallest possible solution to the equation $\tan(e^x) = 0$. (1)

(ii) Copy the diagram and then sketch the curve $y = \tan\left(e^x\right)$ for $\log_e(\frac{\pi}{2}) \langle x \langle \log_e(\frac{3\pi}{2})$. (2)

(iii) How many solutions are there to $\tan(e^x) = 0$ in the domain $1 \langle x \langle 3$? (2)

(iv) Find the equation of the inverse function of $y = \tan(e^x)$ for the case where $x \langle \log_e(\frac{\pi}{2})$ and draw a neat sketch of this curve. (3)

QUESTION 3: (Use a separate Writing Booklet)

(a) The complex number z is given by $z = 1 + \dfrac{1+i}{1-i}$.

(i) Express z in the form $a+ib$, where $a \& b$ are real. (1)

(ii) Find

(α) $\mathrm{Re}(z^2)$. (1)

(β) $|z|$ *and* $\arg(z)$. (1)

(γ) z^5 in the form $x+iy$, where $x \& y$ are real. (2)

(b)

(i) On an Argand diagram sketch the locus of $|z|=1$ & $|z-1|=1$ (1)

(ii) Hence, or otherwise, find in the form $a+ib$, where $a \& b$ are real, all complex numbers simultaneously satisfying $|z|=1$ & $|z-1|=1$. (2)

(c)

(i) Solve $z^3 - 1 = 0$ giving your answers in modulus-argument form. (1)

(ii) Let ω be one of the non-real roots of $z^3 - 1 = 0$.

(α) Show that $1+\omega+\omega^2 = 0$ (1)

(β) Hence simplify $(1+\omega)^5$ (1)

(d)

(i) Find the Cartesian equation and sketch the locus of z if $|z-i| = \mathrm{Im}(z)$ (2)

(ii) What is the least value of $\arg(z)$ in part (i)? (2)

QUESTION 4: (Use a separate Writing Booklet)

(a) If $2x^4+9x^3+6x^2-20x-24=0$ has a triple root, find all the roots. (4)

(b) α,β,γ are the roots of $2x^3-4x^2-3x-1=0$.

(i) Show that $(\alpha-1)(\beta-1)(\gamma-1)=3$ (2)

(ii) Hence find the value of $(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)(\alpha+\beta-\gamma)$ (2)

(c)

(i) Given that $z=\cos\theta+i\sin\theta$, use De Moivre's theorem to show that:

$$z^n+z^{-n}=2\cos n\theta$$ (2)

(ii) Hence, or otherwise, solve the equation

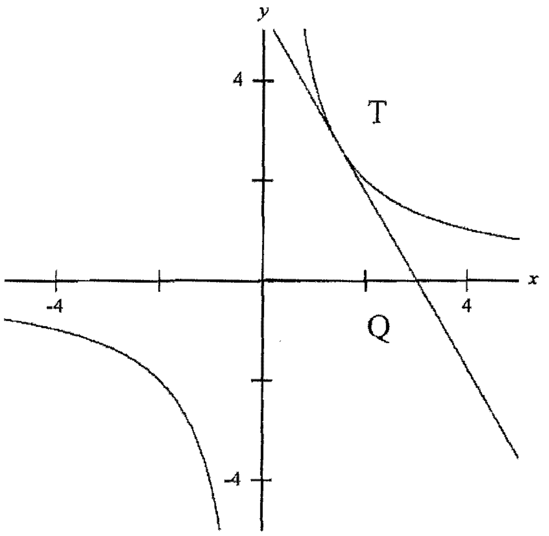
$$2z^4+3z^3+5z^2+3z+2=0$$ (5)

QUESTION 5: (Use a separate Writing Booklet)

(a) The hyperbola H has equation $9x^2 - 16y^2 = 144$. Find the eccentricity, the co-ordinates of its foci, the equation of each directrix and the equation of each asymptote. Sketch the curve and indicate the foci, directrices and asymptotes. (5)

(b) An ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Given that $x = 4\cos\theta$ & $y = 3\sin\theta$ are parametric equations of E, derive the equations of the tangent and normal to the ellipse when $\theta = \frac{\pi}{3}$. (5)

(c) The tangent to the rectangular hyperbola $xy = 4$ at the point T $(2t, \frac{2}{t})$ has equation $x + t^2 y = 4t$. The tangent cuts the x-axis at Q.

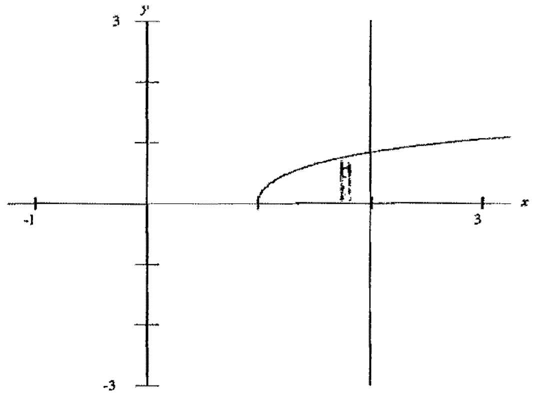


(i) Show that the line through Q, which is perpendicular to the tangent at T, has equation $t^2 x - y = 4t^3$. (2)

(ii) This line cuts the rectangular hyperbola at the points R and S. Find the locus of M, the midpoint of RS, in Cartesian form. (3)

QUESTION 6: (Use a separate Writing Booklet)

(a) By taking slices perpendicular to the x-axis find the volume obtained by rotating the region bounded by the curve $y=\sqrt{\log_e x}$ and the line $x=2$ about the x-axis. (4)



$x=2$

(b)

(i) The region bounded by the curve $y=(x-1)^2$ and the x and y-axes is rotated through 360^0 about the line $y=-\frac{1}{2}$ to form a solid. If a vertical line segment is drawn from the point $P(x,y)$ on the curve, where $0 < x < 1$, to the x-axis it sweeps out an annulus. Show that the area of the annulus is given by:

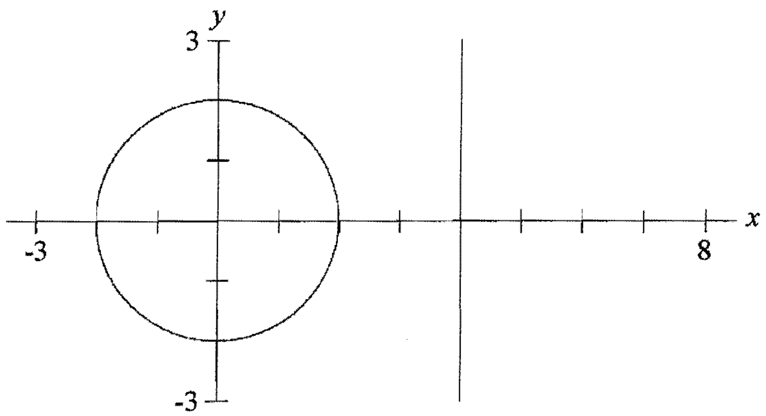
$$A=\pi\left[(x-1)^4+(x-1)^2\right].$$ (3)

(ii) Hence find the volume of the solid. (2)

(c)

(i) Use the substitution $x=2\sin\theta$ to evaluate $\int_{-2}^{2}\sqrt{4-x^2}\,dx$ (3)

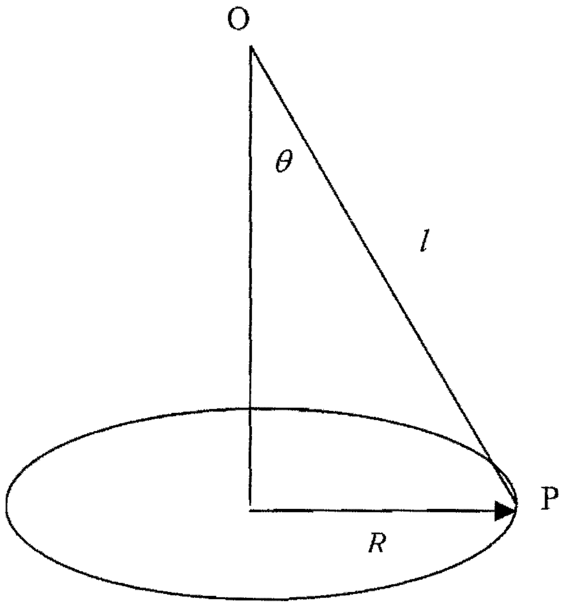
(ii) Find the volume of the solid formed by rotating the circle $x^2+y^2=4$ about the line $x=4$ using the method of cylindrical shells. (4)



$x = 4$

QUESTION 7: (Use a separate Writing Booklet)

(a) A particle P of mass m kg is suspended from the end of a light inelastic string of length l metres which is fixed at a point O. The particle is moving with constant angular velocity ω and describes a circle of radius R metres in the horizontal plane.

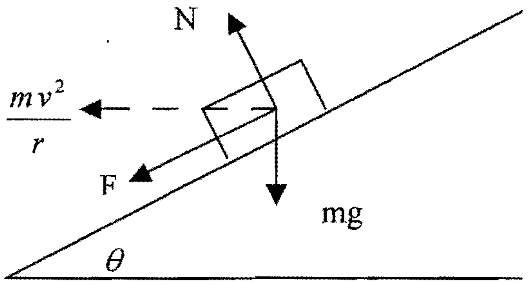


(i) By considering the forces acting on the particle P show that the angle θ between the string and the vertical through O is given by $\tan\theta = \dfrac{R\omega^2}{g}$. (2)

(ii) Show that the period of rotation is given by $2\pi\sqrt{\dfrac{l\cos\theta}{g}}$ (2)

(iii) A particle of mass 600 grams is attached to a light inelastic string, fixed at O, and moves uniformly in a horizontal circle with a period of 1.7 seconds. If the tension in the string is 20 newtons find the length of the string correct to 2 decimal places given that $g = 9.8 m s^{-2}$. (3)

(b) A train line is banked at an angle of θ to the horizontal as shown below.



A train of mass m is travelling at a constant speed v in a horizontal circular arc of radius r on the banked train line.

(i) If the force due to gravity is mg and the force of circular motion is given as $\frac{mv^2}{r}$ find the frictional force F and the normal reaction force N in terms of m, v, r, g and θ. (4)

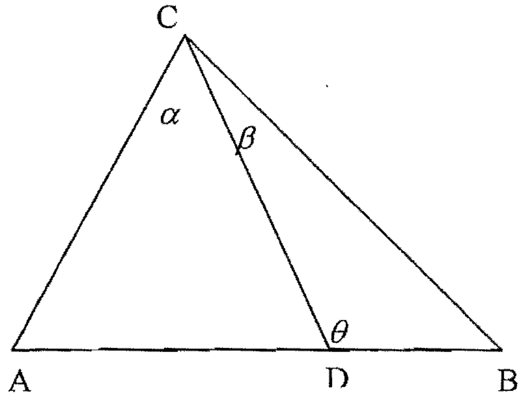
(ii) A train rounding a banked circular bend of radius 500 metres exerts the same frictional force along the slope when travelling at 30 km/h as it does when travelling at 90 km/h but in opposite directions. Assume that acceleration due to gravity g is approximately 10 $m\,s^{-2}$.

(α) Find the angle θ at which the train line is banked correct to the nearest minute. (3)

(β) Find the optimum speed that the train should travel at so that F = 0. (Give your answer correct to the nearest km/h.) (1)

QUESTION 8: (Use a separate Writing Booklet)

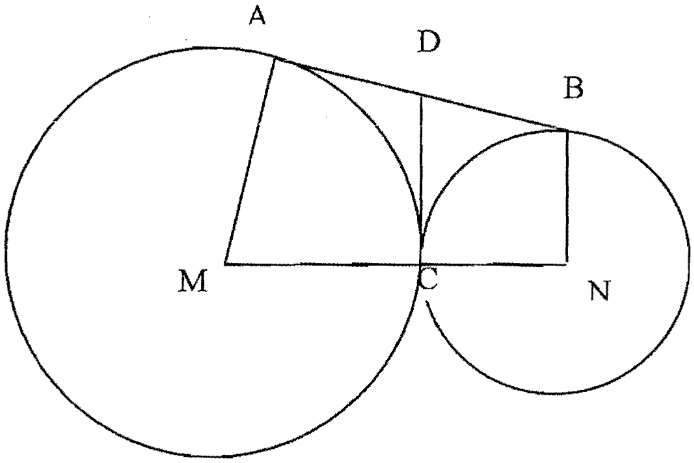
(a)



In ΔABC, D is the point on AB that divides AB internally in the ratio $m:n$. If $\angle ACD=\alpha, \angle BCD=\beta$ and $\angle CDB=\theta$, by using the sine rule in each of the triangles CAD and CDB, show that

$$\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta}=\frac{m}{n}.$$ (4)

(b)



In the diagram MCN is a straight line. Circles are drawn with centres M and N and radii MC and NC respectively. AB is a common tangent to the two circles with points of contact at A and B respectively. CD is a common tangent at C and meets AB at D.

(i) Show that AMCD and BNCD are cyclic quadrilaterals. (2)

(ii) Prove that ΔACD is similar to ΔCBN. (3)

(iii) Prove that MD // CB. (1)

(c)

(i) By finding the equation of the tangent to $y=\log_e x$ at the point where $x=1$ prove that for $x > 0$, $\log_e x \le x-1$ (3)

(ii) If $\{p_1, p_2, \ldots, p_n\}$ is a set of n positive numbers adding to unity, i.e.

$p_1+p_2+\ldots+p_n=1$ and each $p_r > 0$, prove that $\sum_{r=1}^{n} \log_e(n p_r) \le 0$. (2)

END OF PAPER.

2009 C.H.S Ext 2 Trial HSC Solutions

Q1

a) $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx$

Let $u = \sqrt{2x+1}$ $\qquad u^2 = 2x+1$

$= (2x+1)^{1/2}$ $\qquad u^2 - 1 = 2x$

$du = \frac{1}{2}(2x+1)^{-1/2} \cdot 2\,dx$ $\qquad x = \frac{u^2-1}{2}$

ie $du = \dfrac{1}{\sqrt{2x+1}}\,dx$ $\qquad$ When $x=0$, $u=1$; When $x=4$, $u=3$.

$\therefore I = \displaystyle\int_1^3 \frac{u^2-1}{2}\,du$

$= \frac{1}{2}\left[\frac{u^3}{3} - u\right]_1^3$ $\qquad$ (3)

$= \frac{1}{2}\left[\left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)\right]$

$= \frac{1}{2} \times 6\frac{2}{3}$

$= 3\frac{1}{3}$.

b) (i) Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

If $x = \frac{\pi}{2} - u$ $\qquad$ When $x=0$, $u = \frac{\pi}{2}$

$dx = -du$ $\qquad$ When $x = \frac{\pi}{2}$, $u = 0$

$\therefore I = \displaystyle\int_{\pi/2}^{0} \frac{\sin(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)} \cdot -du$

$= -\displaystyle\int_{\pi/2}^{0} \frac{\cos u}{\cos u + \sin u}\,du$

$= \displaystyle\int_0^{\pi/2} \frac{\cos u}{\cos u + \sin u}\,du$ $\qquad$ (2)

$= \displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$

(ii) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$

$= \displaystyle\int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$

$= \displaystyle\int_0^{\pi/2} 1\,dx$

$= \left[x\right]_0^{\pi/2}$ $\qquad$ (2)

$= \frac{\pi}{2}$

$\therefore$ Since $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$

$2I = \frac{\pi}{2}$

$I = \frac{\pi}{4}$.

c) (i) If $\dfrac{8}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$

$8 = A(x^2+4) + (Bx+C)(x+2)$

Let $x = -2$,

$8 = A(8) + 0$

$A = 1$.

Let $x = 0$,

$8 = 4A + 2C$

$8 = 4 + 2C$

$\therefore C = 2$.

Let $x = 1$,

$8 = 5A + (B+2)3$

$8 = 5 + 3B + 6$

$\therefore B = -1$

(2)

$$\therefore \frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}.$$

ii) $$\int_0^2 \frac{8}{(x+2)(x^2+4)}\,dx = \int_0^2 \frac{1}{x+2} + \frac{-x}{x^2+4} + \frac{2}{x^2+4}\,dx$$

$$= \left[\ln(x+2) - \tfrac{1}{2}\ln(x^2+4) + \tan^{-1}\left(\tfrac{x}{2}\right)\right]_0^2$$

$$= \left[\left(\ln 4 - \tfrac{1}{2}\ln 8 + \tfrac{\pi}{4}\right) - \left(\ln 2 - \tfrac{1}{2}\ln 4 + 0\right)\right]$$

$$= 2\ln 2 - \tfrac{3}{2}\ln 2 + \tfrac{\pi}{4} - \ln 2 + \ln 2 - 0$$

$$= \tfrac{1}{2}\ln 2 + \tfrac{\pi}{4}.$$

(2)

d) (i) $$I_0 = \int_0^1 e^x\,dx$$

$$= \left[e^x\right]_0^1$$

$$= e - 1.$$

(1)

(ii) $$I_n = \int_0^1 x^n e^x\,dx \qquad u = x^n \quad u' = nx^{n-1}, \quad v' = e^x \quad v = e^x$$

$$\therefore I_n = \left[x^n e^x\right]_0^1 - n\int_0^1 x^{n-1} e^x\,dx$$

$$I_n = 1.e^1 - 0.e^0 - n\,I_{n-1}$$

$$= e - n\,I_{n-1}.$$

(2)

(iii) $$I_3 = e - 3.I_2$$

$$= e - 3\left[e - 2.I_1\right]$$

$$= e - 3\left[e - 2(e - 1.I_0)\right]$$

$$= e - 3\left[e - 2(e - (e-1))\right]$$

$$= e - 3\left[e - 2(e - e + 1)\right]$$

$$= e - 3\left[e - 2\right]$$

$$= e - 3e + 6$$

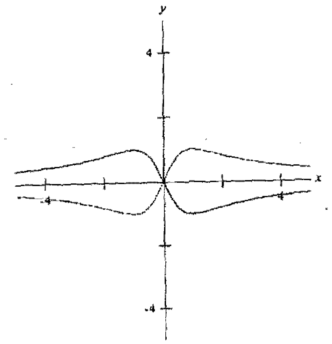
$$= 6 - 2e.$$

(1)

Q2.

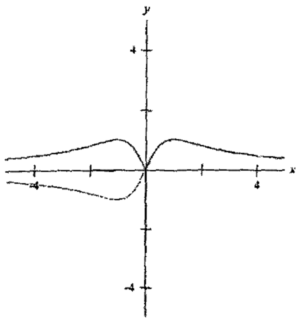
a)

(i)



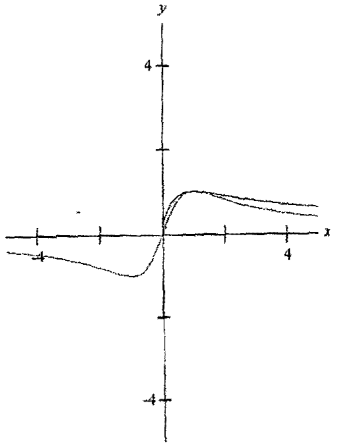
(1)

(ii)



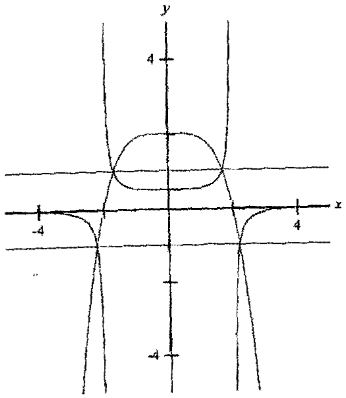
(1)

(iii)



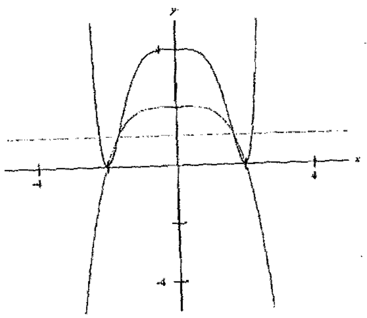
(1)

(iv)



(2)

(v)



(2)

b)

(i) If $\tan(e^x) = 0$

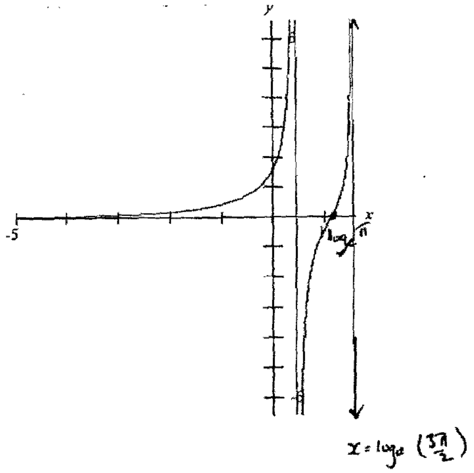
$e^x = 0, \pi, 2\pi, \ldots$

$\therefore$ the smallest positive solution is

$e^x = \pi$

i.e. $x = \log_e \pi$ (1)

(ii)



(2)

(iii) If $\tan(e^x) = 0$

$e^x = 0, \pi, 2\pi, 3\pi, \ldots$

Now $\ln \pi \doteqdot 1.14$, $\ln 6\pi \doteqdot 2.94$, $\ln 7\pi \doteqdot 3.09$

$\therefore x = \log_e \pi, \log_e 2\pi, \ldots \log_e 6\pi$, for $1 < x$

Hence there are 6 solutions. (2)

(iv) $y = \tan(e^x)$ is a one to one function

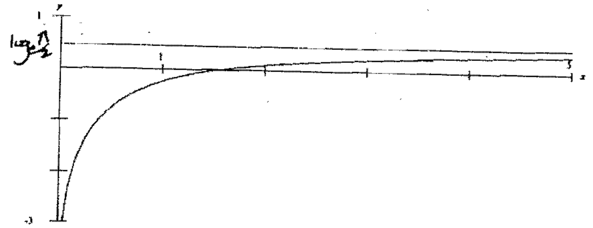
for $x < \log_e\left(\frac{\pi}{2}\right)$

$\therefore$ Its inverse is given by

$x = \tan(e^y)$, $y < \log_e\left(\frac{\pi}{2}\right)$

$\therefore e^y = \tan^{-1}(x)$

$y = \log_e\left(\tan^{-1}(x)\right)$. (2)



(1)

Q3

c)(i) $z = 1 + \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$= 1 + \frac{(1+i)^2}{1-i^2}$

$= 1 + \frac{1+2i+i^2}{2}$

$= \frac{2+1+2i-1}{2}$ (1)

$= 1+i$

(ii) (α) $z^2 = (1+i)^2$

$= 1+2i-1$

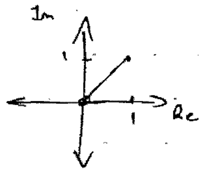
$= 2i$

$\therefore Re(z^2) = 0$ (1)

(β) $|z| = \sqrt{1+1}$

$= \sqrt{2}$

$Arg(z) = \frac{\pi}{4}$.



(1)

(γ) Now $z = \sqrt{2}\,cis\left(\frac{\pi}{4}\right)$

$\therefore z^5 = (\sqrt{2})^5\, cis\left(\frac{5\pi}{4}\right)$

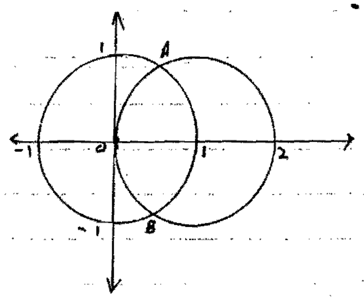
$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$ (2)

$= 4(-1-i)$

$= -4-4i$

b) (i)



(1)

(ii) Let the points of intersection be A & B & let the centres of the circles be O & C

$\triangle AOC$ is equilateral since $OA = OC = AC = 1$ unit

& $\triangle BOC$ " " " $OB = OC = BC = 1$ unit

$\therefore$ A is the pt $cis\left(\frac{\pi}{3}\right)$ & B is the pt $cis\left(-\frac{\pi}{3}\right)$

$\therefore$ the sol^n are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ & $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(2)

c) (i) Let $z = cis\,\theta$

If $z^3 = 1$

$cis\,3\theta = 1$

$3\theta = 0, 2\pi, 4\pi,$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$

(1)

$\therefore \quad z = 1, \; cis\left(\frac{2\pi}{3}\right), \; cis\left(\frac{4\pi}{3}\right)$

(ii) Let the roots be $1, \omega, \omega^2$

For $z^3 - 1 = 0$

$\Sigma\alpha : 1 + \omega + \omega^2 = -\frac{b}{a}$

$= 0$

(1)

b) $(1+\omega)^5 = (-\omega^2)^5$

$= -(\omega)^{10}$

$= (\omega^3)^3 \times -\omega$

Since ω is a solution to $z^3 - 1 = 0$

$\omega^3 = 1.$

(1)

$\therefore (1+\omega)^5 = 1^3 \times -\omega$

$= -\omega.$

d) Let $z = x + iy$

$|z - i| = |x + i(y-1)|$

$= \sqrt{x^2 + (y-1)^2}$

$Im(z) = y$

$\therefore \quad y = \sqrt{x^2 + (y-1)^2}$

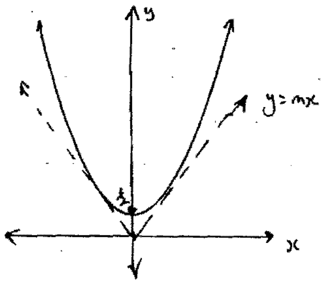
$y^2 = x^2 + (y-1)^2$

$y^2 = x^2 + y^2 - 2y + 1$

$\therefore \quad 2y = x^2 + 1$

$y = \frac{x^2 + 1}{2}.$

(2)



(ii) Let $y = mx$ be a tangent to $y = \frac{x^2+1}{2}$

Pt of intersection is given by

$$\frac{x^2+1}{2} = mx$$

$$x^2 - 2mx + 1 = 0$$

If $y = mx$ is a tangent $\Delta = 0$

$$\text{ie } 4m^2 - 4 = 0$$
$$m^2 = 1$$
$$m = \pm 1$$

(2)

$\therefore$ the least value of $\arg(z)$ is $\tan^{-1}(1)$

ie. $\pi/4$.

Q4

a) Let $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12$

If $P''(x) = 0$

$$24x^2 + 54x + 12 = 0$$
$$4x^2 + 9x + 2 = 0$$
$$(4x+1)(x+2) = 0$$
$$x = -1/4 \text{ or } -2.$$

Now $P'(-1/4) = 8(-1/4)^3 + 27(-1/4)^2 + 12(-1/4) - 20$
$\neq 0.$

But $P'(-2) = 8(-2)^3 + 27(-2)^2 + 12(-2) - 20$
$= 0$

$\therefore -2$ is the triple root.

(4)

The sum of the roots $\sum \alpha = -\frac{b}{a}$

$$\therefore -2 + -2 + -2 + k = -\frac{9}{2}$$
$$-6 + k = -\frac{9}{2}$$
$$k = \frac{3}{2}$$

$\therefore$ The roots are $-2, -2, -2$ & $\frac{3}{2}$

b) (i) $(\alpha-1)(\beta-1)(\gamma-1) = (\alpha-1)[\beta\gamma - \beta - \gamma + 1]$

$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$

$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1$

$= -\frac{d}{a} - \frac{c}{a} + -\frac{b}{a} - 1$

$= \frac{1}{2} - -\frac{3}{2} + 2 - 1$

$= 3$

(2)

(ii) Since $\alpha+\beta+\gamma = 2$

$\alpha+\beta-\gamma = 2-2\gamma$

Similarly $\beta+\gamma-\alpha = 2-2\alpha$

$\alpha+\gamma-\beta = 2-2\beta$

$\therefore (\beta+\gamma-\alpha)(\gamma+\alpha-\beta)(\alpha+\beta-\gamma)$

$= (2-2\alpha)(2-2\beta)(2-2\gamma)$

$= -2(\alpha-1) \cdot -2(\beta-1) \cdot -2(\gamma-1)$

$= -8(\alpha-1)(\beta-1)(\gamma-1)$

$= -8 \times 3$

$= -24$

(2)

(iii) c) i) If $z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$

(2)

& $z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin(n\theta)$ since $\cos -A = \cos$ & $\sin -A = -\sin$

$\therefore z^n + z^{-n} = 2\cos(n\theta)$

ii) If $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$

$z^2(2z^2 + 3z + 5 + 3z^{-1} + 2z^{-2}) = 0$

$z^2[2z^2 + 2z^{-2} + 3z + 3z^{-1} + 5] = 0$

$z^2[2(2\cos 2\theta) + 3(2\cos\theta) + 5] = 0$

$z^2[4(2\cos^2\theta - 1) + 6\cos\theta + 5] = 0$

$z^2[8\cos^2\theta + 6\cos\theta + 1] = 0$

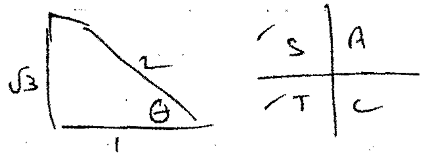
So $z^2[(4\cos\theta + 1)(2\cos\theta + 1)] = 0$

$\therefore z^2 = 0$, $\cos\theta = -\frac{1}{4}$, $\cos\theta = -\frac{1}{2}$

If $\cos\theta = -\frac{1}{4}$

$\sin\theta = \pm\frac{\sqrt{15}}{4}$.

If $\cos\theta = -\frac{1}{2}$



(5)

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

If $z^2 = 0$, $\cos 2\theta + i\sin 2\theta = 0 + 0i$

$\therefore \cos 2\theta = 0$ & $\sin 2\theta = 0$

No soln.

$\therefore z = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Q5.

a) $9x^2 - 16y^2 = 144$

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$\therefore a = 4, \quad b = 3.$

Now $b^2 = a^2(e^2 - 1)$

$$\frac{9}{16} = e^2 - 1$$

$$e^2 = \frac{25}{16}$$

$$e = 5/4$$

The eccentricity is $5/4$.

$$ae = 4 \times \frac{5}{4} = 5$$

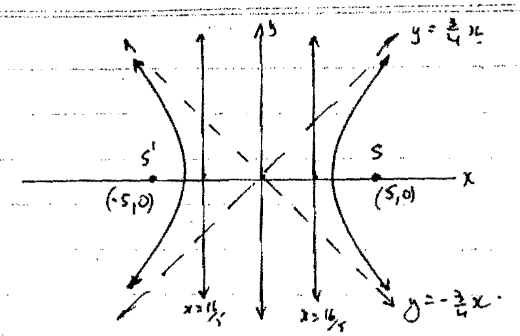
The foci are $(5,0)$ & $(-5,0)$

$$\frac{a}{e} = \frac{4}{5/4} = 16/5$$

The eqn of the directrices are $x = \pm 16/5$

The eqn of the asymptotes are $y = \pm \frac{3}{4}x$.

(4)



(1)

b) If $x = 4\cos\theta$ $\quad \frac{dx}{d\theta} = -4\sin\theta$

If $y = 3\sin\theta$ $\quad \frac{dy}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 3\cos\theta \times \frac{-1}{4\sin\theta}$$

$$= -\frac{3}{4}\cot\theta$$

(2)

When $\theta = \frac{\pi}{3}$,

$\frac{dy}{dx} = -\frac{3}{4} \cdot \cot\frac{\pi}{3}$, $\quad x = 4\cos\frac{\pi}{3}$, $\quad y = 3\sin\frac{\pi}{3}$

$= -\frac{3}{4} \cdot \frac{1}{\sqrt{3}}$ $\quad = 4 \cdot \frac{1}{2}$ $\quad = 3 \cdot \frac{\sqrt{3}}{2}$

$= -\sqrt{3}/4$ $\quad = 2$ $\quad = \frac{3\sqrt{3}}{2}$

$\therefore$ the eqn of the tangent is $y - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{4}(x - 2)$

$$4y - 6\sqrt{3} = -\sqrt{3}x + 2\sqrt{3}$$

i.e. $\sqrt{3}x + 4y - 8\sqrt{3} = 0$ (1)

The eqn of the normal is of the form:

$$4x - \sqrt{3}y = k.$$

So $4(2) - \sqrt{3}\left(\frac{3\sqrt{3}}{2}\right) = k$

$$8 - \frac{9}{2} = k$$
$$k = \frac{7}{2}$$

ie $4x - \sqrt{3}y = \frac{7}{2}$ (1)

or $8x - 2\sqrt{3}y - 14 = 0$.

c) i) If $x + t^2 y = 4t$

when $y = 0$, $x = 4t$

$\therefore$ Q is the pt $(4t, 0)$ (2)

Also $x + t^2 y = 4t$
$$t^2 y = -x + 4t$$
$$y = -\frac{1}{t^2}x + \frac{4}{t}$$

$\therefore$ the grad. of the $\perp$ to TQ is t^2

the eqn of the normal is
$$y - 0 = t^2(x - 4t)$$
$$y = t^2 x - 4t^3$$

ie. $t^2 x - y = 4t^3$

(ii) Pts of intersection occur when

$$t^2 x - 4t^3 = \frac{4}{x}$$

i.e. $t^2 x^2 - 4t^3 x - 4 = 0$ ⊛

If M is the point (X, Y)

$X = \frac{x_1 + x_2}{2}$ where x_1 & x_2 are the roots of ⊛

But $x_1 + x_2 = -\frac{b}{a}$
$$= \frac{4t^3}{t^2}$$
$$= 4t.$$

$\therefore X = 2t$

If $X = 2t$, $y = t^2 x - 4t^3$
$$= t^2 \times 2t - 4 \times t^3$$
$$= 2t^3 - 4t^3$$
$$Y = -2t^3.$$

M is the pt $(2t, -2t^3)$ (3)

If $x = 2t$
$$t = \frac{x}{2}$$

$$y = -2t^3$$
$$= -2\left(\frac{x^3}{8}\right)$$
$$= -\frac{x^3}{4}$$

The locus of M is $y = \frac{-x^3}{4}$.

Q6

a) Let the width of each slice be Δx

$\Delta V = \pi . y^2 . \Delta x$

$\therefore V = \lim_{\Delta x \to 0} \sum_1^2 \pi . y^2 . \Delta x$

$= \int_1^2 \pi y^2 \, dx$

$= \pi \int_1^2 \log_e x \, dx$ (4)

$u = \log_e x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$\therefore I = \pi \left\{ \left[x \log_e x \right]_1^2 - \int_1^2 \frac{1}{x} . x \, dx \right\}$

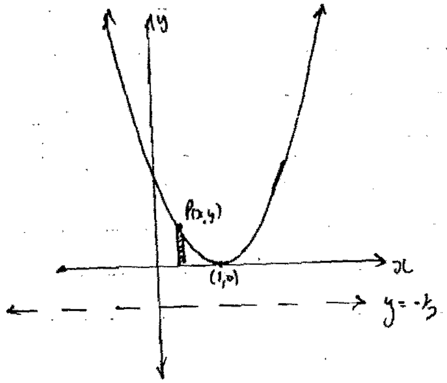
$= \pi \left\{ (2 \log_e 2 - 2 \log_e 1) - \left[x \right]_1^2 \right\}$

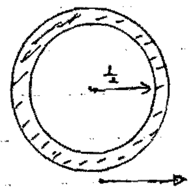
$= \pi (2 \log_e 2 - (2-1) \}$

$= \pi (2 \log_e 2 - 1)$ units^3.

b)

(i)





$A = \pi (R^2 - r^2)$ where $R = y + \frac{1}{2}$, $r = \frac{1}{2}$

$\therefore A = \pi \left[(y+\frac{1}{2})^2 - (\frac{1}{2})^2 \right]$

$= \pi \left[(y + \frac{1}{2} + \frac{1}{2})(y + \frac{1}{2} - \frac{1}{2}) \right]$

$= \pi \left[(y+1) . y \right]$ (3)

$= \pi \left[\{ (x-1)^2 + 1 \} \{ x-1 \}^2 \right]$

$= \pi \left[(x-1)^4 + (x-1)^2 \right]$

(ii) $\Delta V = \pi \left[(x-1)^4 + (x-1)^2 \right] . \Delta x$

$V = \lim_{\Delta x \to 0} \sum_0^1 \pi \left[(x-1)^4 + (x-1)^2 \right] . \Delta x$

$= \pi \int_0^1 (x-1)^4 + (x-1)^2 \, dx$

$= \pi \left[\frac{(x-1)^5}{5} + \frac{(x-1)^3}{3} \right]_0^1$

$= \pi \left[(0+0) - (-\frac{1}{5} + -\frac{1}{3}) \right]$ (2)

$= \pi \times \frac{8}{15}$

$= \frac{8\pi}{15}$ units^3.

c)

(i) If $x = 2 \sin\theta$ When $x = -2$, $\theta = -\frac{\pi}{2}$

$dx = 2 \cos\theta \, d\theta$ When $x = 2$, $\theta = \frac{\pi}{2}$

$\therefore I = \int_{-\pi/2}^{\pi/2} \sqrt{4 - 4\sin^2\theta} . 2 \cos\theta \, d\theta$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{4\cos^2\theta} \,.\, 2\cos\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2\cos\theta \,.\, 2\cos\theta \, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta \, d\theta$$

$$= 4\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta \right]_{-\pi/2}^{\pi/2}$$

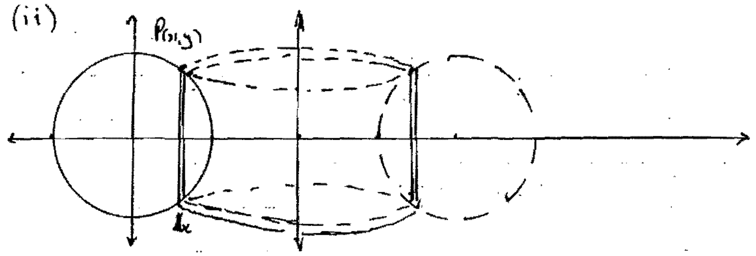
$$= 4\left\{ \left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right) - \left(-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi)\right) \right\}$$

$$= 4\left(\frac{\pi}{4} + 0 + \frac{\pi}{4} - 0 \right)$$

$$= 2\pi$$

(3)

(ii)



The radius of the inner shell is $4 - x + \Delta x$

" " " " outer " " $4-x$

" height " " shells is $2y$

$$\Delta V = \pi\,(R^2 - r^2).h$$

$$= \pi\,(R+r)(R-r).h$$

$$= \pi\left[(4-x+\Delta x) + (4-x)\right]\left[(4-x+\Delta x) - (4-x)\right].2y.$$

$$= \pi\left[2(4-x) + \Delta x\right].\Delta x.2y.$$

$$\therefore V = \lim_{\Delta x \to 0} \sum_{-2}^{2} \pi\left[2(4-x) + \Delta x\right].\Delta x.2y$$

$$= 4\pi \int_{-2}^{2} (4-x).\sqrt{4-x^2}\,dx$$

$$= 4\pi \int_{-2}^{2} 4\sqrt{4-x^2} - x\sqrt{4-x^2}\,dx$$

$$= 16\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx - 4\pi\int_{-2}^{2} x\sqrt{4-x^2}$$

Now if $f(x) = x\sqrt{4-x^2}$

$f(-x) = -x\sqrt{4-x^2}$

$= -f(x)$

$\therefore f(x)$ is odd

Hence $\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$

$$\therefore V = 16\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 16\pi \times 2\pi$$

$$= 32\pi \text{ units}^3.$$

(4)

$$\text{Period} = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{\frac{g}{l\cos\theta}}}$$ (2)

$$= 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

(iii) Since $T\cos\theta = mg$ — $g = 9.8$, $m = 0.6\text{kg}$.

$$20\cos\theta = 0.6 \times 9.8$$

$$\cos\theta = \frac{0.6 \times 9.8}{20}$$

$$\therefore\ 1.7 = 2\pi\sqrt{\frac{l\left(\frac{0.6\times 9.8}{20}\right)}{9.8}}$$

$$1.7 = 2\pi\sqrt{l\left(\frac{0.6}{20}\right)}$$ (3)

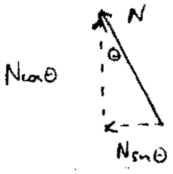
$$\left(\frac{1.7}{2\pi}\right)^2 = l\left(\frac{0.6}{20}\right)$$

$$l = \frac{\left(\frac{1.7}{2\pi}\right)^2}{0.03}$$

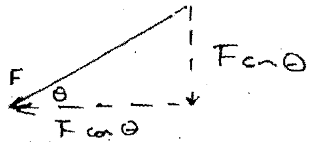
$$\doteqdot 2.44 \text{ metres } (2.d.p)$$

b) (1) Forces acting on the train

N:

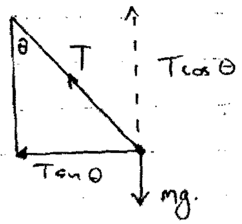


F:



Q7

(a) (i) Let T be the tension in the string



Resolving forces vertically and horizontally

V: $T\cos\theta - mg = 0$

H: $T\sin\theta = mR\omega^2$.

$\therefore\ T\cos\theta = mg$ & $T\sin\theta = mR\omega^2$ (2)

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mR\omega^2}{mg}$$

$$\tan\theta = \frac{R\omega^2}{g}$$

(ii) If $\tan\theta = \frac{R\omega^2}{g}$

$$\omega^2 = \frac{g\tan\theta}{R}$$

But $\frac{R}{l} = \sin\theta$

$$\therefore\ \omega^2 = \frac{g\tan\theta}{l\sin\theta}$$

$$= \frac{g\frac{\sin\theta}{\cos\theta}}{l\sin\theta}$$

$$= \frac{g}{l\cos\theta}$$

Resolving forces vertically & horizontally

$$H: \quad N\sin\theta + F\cos\theta = \frac{mv^2}{r} \quad \text{— ①}$$

$$V: \quad N\cos\theta - F\sin\theta = mg \quad \text{— ②}$$

① × cos θ & ② × sin θ

$$N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta \quad \text{— ③}$$
$$N\sin\theta\cos\theta - F\sin^2\theta = mg\sin\theta \quad \text{— ④}$$

③ − ④ gives

$$F\cos^2\theta + F\sin^2\theta = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$

$$\therefore F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$ (2)

① × sin θ + ② × cos θ

$$N\sin^2\theta + F\sin\theta\cos\theta = \frac{mv^2}{r}\sin\theta \quad \text{— ⑤}$$
$$N\cos^2\theta - F\sin\theta\cos\theta = mg\cos\theta \quad \text{— ⑥}$$

⑤ + ⑥ gives

$$N\sin^2\theta + N\cos^2\theta = \frac{mv^2}{r}\sin\theta + mg\cos\theta$$

$$\therefore N = \frac{mv^2}{r}\sin\theta + mg\cos\theta$$ (2)

(ii)

(α) $30\text{km/hr} = 30 \times 1000 \div 60 \div 60 \text{ ms}^{-1}$
$= \frac{25}{3} \text{ ms}^{-1}$

$\therefore 90\text{km/hr} = \frac{75}{3} \text{ ms}^{-1}$

Let the frictional force at 30km/hr be F_1
" " " " " 90km/hr " F_2

$$F_1 = \frac{m\left(\frac{25}{3}\right)^2\cos\theta}{500} - m.10.\sin\theta$$

$$F_2 = \frac{m\left(\frac{75}{3}\right)^2\cos\theta}{500} - m.10.\sin\theta$$

$$F_1 + F_2 = 0$$

$$\therefore m\cos\theta\left[\frac{\left(\frac{75}{3}\right)^2 + \left(\frac{25}{3}\right)^2}{500}\right] - 20m\sin\theta = 0$$

$$\therefore 20\tan\theta = \frac{\left(\frac{75}{3}\right)^2 + \left(\frac{25}{3}\right)^2}{500}$$ (3)

$$\tan\theta = \frac{\left(\frac{75}{3}\right)^2 + \left(\frac{25}{3}\right)^2}{10000}$$

$\theta \doteqdot 3°58'$ nearest min.

(β) if $F = 0$

$$\frac{mv^2}{r}\cos\theta = mg\sin\theta$$

$$\tan\theta = \frac{v^2}{rg}$$ (1)

$$v^2 = rg\tan\theta$$

$$v^2 = 500 \times 10 \times \tan 3°58'$$
$v \doteqdot 18.62 \text{ ms}^{-1}$
$\doteqdot 67 \text{ km/hr}$ (nearest km/hr)

Q8

a) In ΔCAD, $\angle CAD = \theta - \alpha$ (ext. $\angle$ of a Δ theorem)

$$\therefore \frac{CD}{\sin(\theta-\alpha)} = \frac{AD}{\sin\alpha}$$

$$AD = \frac{CD \sin\alpha}{\sin(\theta-\alpha)}$$

In ΔCDB, $\angle CBD = 180 - (\theta+\beta)$ ($\angle$ sum of a Δ is 180°)

$$\therefore \frac{CD}{\sin[180-(\theta+\beta)]} = \frac{DB}{\sin\beta}.$$

$$DB = \frac{CD\sin\beta}{\sin[180-(\theta+\beta)]}$$

$$= \frac{CD \sin\beta}{\sin(\theta+\beta)} \quad \text{since } \sin(180-A) = \sin A.$$

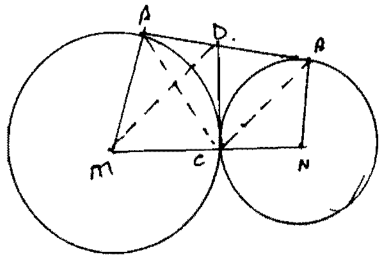
$$\therefore \frac{AD}{BD} = \frac{\frac{CD\sin\alpha}{\sin(\theta-\alpha)}}{\frac{CD\sin\beta}{\sin(\theta+\beta)}}$$

(4)

$$\therefore \frac{m}{n} = \frac{\cancel{CD}\sin\alpha}{\sin(\theta-\alpha)} \times \frac{\sin(\theta+\beta)}{\cancel{CD}\sin\beta}$$

$$\text{ie. } \frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{m}{n}$$

b)



(i) $\angle DAM = \angle DCM = 90^\circ$ (tangent $\perp$ radius at pt of contact)
& $\angle DBN = \angle DCN = 90^\circ$ (" " " " " ")
$\therefore \angle DAM$ & $\angle DCM$ form a pair of suppl. opp. $\angle$'s
& $\angle DBN$ & $\angle DCN$ " " " " " "
Hence AMCD & BNCD are cyclic quads.

(2)

(ii) Join A to C & B to C

Let $\angle ADC$ be θ
$\therefore \angle BNC = \theta$ (ext. $\angle$ of a cyclic quad theorem)

Now $AD = DC$ (tangents from an external pt. are equal in length)
$\therefore \Delta ADC$ is isosceles.

$\therefore \angle DAC, \angle DCA = 90 - \frac{\theta}{2}$ ($\angle$ sum of a Δ is 180° & base $\angle$'s of an isosc Δ are equal)

Also $BN = CN$ (equal radii)
$\therefore \Delta BNC$ is isosceles

(3)

$\therefore \angle NCB = \angle NBC$ ($\angle$ sum of a Δ is 180° & base $\angle$'s of an isosc. Δ are equal)
$\therefore \Delta ACD \,|||\, \Delta CBN$ (Δ's are equiangular)

(iii) Join M to D
Since ADMC is a cyclic quad.
$\angle CMD = \angle CAD = 90 - \frac{\theta}{2}$ ($\angle$'s in the same segment are equal)
$\therefore \angle CMD$ & $\angle NCB$ form a pair of equal corresp. $\angle$'s.

(1)

Hence $MD \parallel CB$

c) (i) If $y = \log_e x$

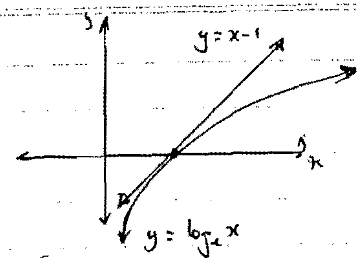
$y' = \frac{1}{x}$

When $x = 1$, $y' = 1$, $y = 0$

$\therefore$ The eqn of the tangent at $(1, 0)$ is

$y - 0 = 1(x-1)$

$y = x - 1$



(3)

Since $y = x - 1$ lies above $y = \log_e x$

$\log_e x \le x - 1$ with equality when $x = 1$

(ii) Since $p_r > 0$

$\log_e (np_1) \le np_1 - 1$

$\log_e (np_2) \le np_2 - 1$

$\vdots$

$\log_e (np_n) \le np_n - 1$

Adding these we get

$$\sum_{r=1}^{n} \log_e (np_r) \le np_1 - 1 + np_2 - 1 + \dots + np_n - 1$$

$$= n(p_1 + p_2 + \dots + p_n) - 1 \times n$$

$$= n \times 1 - n$$

$$= 0.$$

$$\therefore \sum_{r=1}^{n} \log_e (np_r) \le 0.$$

(2)